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Algebraic Constraint Quantization and the Pseudo–Rigid Body*

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Abstract

The pseudo–rigid body represents an example of a constrained system with a non-unimodular gauge group. This system is used as a testing ground for the application of an algebraic constraint quantization scheme which focusses on observable quantities, translating the vanishing of the constraints into representation conditions on the algebra of observables. The constraint which is responsible for the non-unimodularity of the gauge group is shown not to contribute to the observable content of the constraints, *i.e.*, not to impose any restrictions on the construction of the quantum theory of the system. The application of the algebraic constraint quantization scheme yields a unique quantization of the physical degrees of freedom, which are shown to form a realization of the so-called $CM(N)$–model of collective motions.

I. Introduction

The pseudo–rigid body [1] represents an example of a first class constrained system with a complicated, non–Abelian and non-unimodular gauge group. In the present paper this system will be used a testing ground for the application of an algebraic concept for the implementation of classical phase space constraints into the quantum theory, formulated heuristically in Ref. [2]. The aim of this algebraic concept is to circumvent the technical and conceptual problems which beset the currently used methods for the quantization of constrained systems, where one has to impose requirements upon the "quantization" of unphysical quantities, like the constraints or gauge conditions, which are subsequently used to project the physical states out of an extended Hilbert space. In contrast, as the connection between the classical and quantum descriptions of a physical system is closest on the algebraic level, the central idea of the algebraic concept consists in translating the "vanishing" of the constraints into conditions which are imposed upon observable quantities. This is achieved by treating the intrinsically defined observable content of the constraints (see below) as supplying representation conditions for the identification of the physical representation of the algebra of observables.

The example of the pseudo-rigid body has been chosen because Duval, Elhadad, Gotay, Śniatycki, and Tuynman [3, 4] have shown that, when quantizing a first class constrained system with a non-unimodular gauge group H using the Dirac quantization procedure, the usual invariance condition for projecting the physical states out of an extended Hilbert space

$$\hat{\mathsf{J}}_\xi \Psi_{phys} = 0 \tag{1}$$

must be replaced by a condition of quasi–invariance

$$\hat{\mathsf{J}}_\xi \Psi_{phys} = -\frac{i}{2}\mathrm{tr}(\mathrm{ad}_\xi) \cdot \Psi_{phys} \tag{2}$$

($\xi \in \mathcal{L}H$, the Lie algebra of H; $\hat{\mathsf{J}}_\xi$ is the operator corresponding to the generator J_ξ of the action of the one-parameter subgroup of H generated by ξ). This means that the naive application to constrained systems of, *e.g.*, the Dirac quantization scheme, which *does* require the quantization of unobservable quantities, *viz.* the constraints, can lead to "demonstrably erroneous conclusions" [1].

Therefore, this system presents a touchstone for the algebraic concept for the quantization of constrained systems. It will be shown that its application reproduces, in particular, the results of Ref. [1]. In contrast to Ref. [1], our method can be applied without having to take special care of the non-unimodularity of the gauge group: the constraint which causes the non-unimodularity does not possess any observable content. Nevertheless, the discussion of this example gives rise to a more precise and more widely applicable formulation of the concept.

The plan of the paper is as follows. In Sec. II the algebraic concept for the implementation of classical phase space constraints into the quantum theory is formulated. In Sec. III the pseudo-rigid body is introduced in an arbitrary number N of space dimensions. The identification of the algebra of observables leads to the *CM(N)*–model of collective motions. In Sec. IV the discussion is specialized to the dimensions $N = 2$ and $N = 3$, the observable content of the constraints is determined, and a short description of the free dynamics is given. Finally, the quantization of the system according to the algebraic concept, carried out in Sec. V, yields a unique identification of the physical representation of the algebra of observables.

II. Algebraic constraint quantization

A. Heuristic considerations

To begin with, quantization is understood as the construction of the quantum algebra of observables, starting from the classical algebra of observables, and the identification of that irreducible $*$–representation of it, which provides the description of the physical system in question. The classical respectively quantum algebra of observables is a (graded, involutive) Poisson respectively commutator algebra which is generated polynomially by a set of *fundamental* observables. The physical representation of the quantum algebra of observables, as a commutator algebra, is distinguished by additional algebraic structures, like characteristic relations (with respect to the associative product) between its elements, the values of its Casimir elements, or extremal properties. Likewise, the physical realization of the classical algebra of observables, as a Poisson algebra, is distinguished by algebraic structures which correspond to those of the quantum algebra of observables. Strictly speaking, the set of additional algebraic structures, required to determine its physical realization, should be considered as part of the definition of the algebra of observables; it should be chosen minimally, such that there is just one faithful representation of the so defined algebra of observables.

Now the question arises, how the construction of the quantum theory can be affected by the presence of constraints, and how the condition of the “vanishing” of the constraints, *i.e.* the gauge invariance of the system, can be implemented into the quantum theory [5].

The first place, where the constraints could gain influence on the construction of the quantum theory is the characterization of the algebra of observables by the commutators of its elements. This influence can be excluded by requiring that the set of fundamental observables, which generates the classical algebra of observables, consists of proper observables, *i.e.* that the fundamental observables are gauge *in*variant and that their linear span does not contain any generators of pure gauge

transformations. For, in that case, assuming for the moment that the reduced phase space does exist, each element of the set of fundamental observables induces a non-vanishing function on the reduced phase space and the abstract Poisson algebras, generated polynomially by these two sets of functions, are isomorphic. Consequently, the two realizations of this algebra can only differ as regards the set of algebraic structures which are needed to characterize the algebra of observables beyond the commutation relations, and the differences must disappear upon the vanishing of the constraints. For this to be the case, there must exist dependencies between those elements of the algebra of observables, in terms of which these algebraic structures are formulated, and certain gauge invariant combinations of the constraints (*e.g.* functional dependencies, which, upon the vanishing of the constraints, induce relations between the elements of the algebra of observables). So, the only possibility for the constraints to work their way into the quantum theory is the existence of such dependencies, which represent the remaining gauge redundancy that has not been eliminated by the choice of the set of fundamental observables.

In the present work it will be assumed that the representations of the quantum algebra of observables can be characterized by the values of its Casimir invariants alone [7] (otherwise further algebraic structures must be treated in essentially the same way as the identities for the Casimirs are treated below). This is usually the case in physically relevant systems, especially if the fundamental observables form a Lie algebra.

On this assumption the constraints can only have an impact on the construction of the quantum theory if they impose restrictions on the values of the Casimirs of the algebra of observables. That is, there must exist functional dependencies, in the classical theory, which allow to identify certain gauge invariant combinations of the constraints with Casimir elements of the algebra of observables, and the condition of the vanishing of the constraints induces identities which have to be fulfilled by the Casimir elements of the algebra of observables. These identities allow to translate the gauge invariance of the classical system into conditions on observable quantities, which will be referred to as the *observable content of the constraints* [8]. By imposing correspondence requirements, the operator versions of the said gauge invariant combinations of the constraints can be identified with central elements of the quantum algebra of observables, which permit to formulate the observable content of the constraints, and thus to implement the constraints, on the level of quantum theory.

B. Formulation of the concept

In the following the most important steps for the realization of the algebraic concept for the implementation of classical phase space constraints into the quantum theory will be enumerated. This enumeration should not be misunderstood as a

quantization program that can be applied algorithmically. Rather, it is meant as a statement of the principles which, in one form or the other, should apply to the quantization of an arbitrary constrained system (with the restriction stated above), but which has to be adapted in a case by case analysis to the concrete situation.

In any case the starting point is a gauge invariant Hamiltonian system with phase space P and a set of first–class constraints C_i. The following notation will be employed:
$\mathcal{F} = C^\infty(P)$ is the set of all smooth functions on P;
$\mathcal{C} = \{f \in \mathcal{F} | \exists g^i \in \mathcal{F} : f = \sum_i g^i \mathsf{C}_i\}$ is the set of all weakly vanishing functions on P (under suitable regularity assumptions on the constraints C_i, cf. [9]);
$\mathcal{P} = \{f \in \mathcal{F} | \{f, \mathsf{C}_i\} = 0 \ \forall i\}$ is the strong Poisson commutant of the constraints;
$\mathcal{G} = \mathcal{C} \cap \mathcal{P}$ is the set of gauge invariant combinations of the constraints; the elements of $\mathcal{G}$ will be referred to as the *generalized Casimir elements of the constraints*.

- Observables: Choose a set $\tilde{\mathcal{O}} \subset \mathcal{P} \setminus \mathcal{G}$ of fundamental observables, such that $L(\tilde{\mathcal{O}}) \cap \mathcal{G} = \{0\}$ (where $L(\tilde{\mathcal{O}})$ is the linear span of the elements of $\tilde{\mathcal{O}}$). $\tilde{\mathcal{O}}$ must generate $\mathcal{P}$ weakly, *i.e.* the (closure, with respect to a suitable norm, of the) polynomial algebra over $\tilde{\mathcal{O}}$ must coincide with $\mathcal{P}$ up to equivalence ($\mathcal{P} \ni f \approx g \in \mathcal{P} \iff f - g \in \mathcal{G}$), at least locally, cf. [10]. If there is a $*$–involution on $\mathcal{P}$, $\tilde{\mathcal{O}}$ has to be closed with respect to it.

  Equipped with the Poisson bracket, $L(\tilde{\mathcal{O}})$ should possibly form a Lie algebra (which, in that case, will also be denoted by $\tilde{\mathcal{O}}$). Otherwise the Poisson brackets of the elements of $\tilde{\mathcal{O}}$ must be polynomial in the fundamental observables. The classical algebra of observables $\mathcal{O}$ is the Poisson algebra generated polynomially by $\tilde{\mathcal{O}}$ (or, more generally, its completion with respect to a suitable norm).

  The generators of the symmetry algebra $\mathcal{S}$ of the system should be contained in $\tilde{\mathcal{O}}$ [11], and the Hamiltonian must be a simple polynomial function of the elements of $\tilde{\mathcal{O}}$.

- The Observable Content of the Constraints: Determine the observable content of the constraints, *i.e.* the functional dependencies between the Casimir elements of $\mathcal{O}$ and the constraints, and the conditions which are imposed upon the Casimir elements of $\mathcal{O}$ by the vanishing of the constraints.

  The set of generalized Casimir elements of the constraints, respectively of the corresponding Casimir elements of $\mathcal{O}$, which enter into the functional dependencies, will be denoted by $\mathcal{OC}$.

- Identities: In addition, one has to determine the identities for the Casimir elements of $\mathcal{O}$ which do not involve the constraints.

- Quantum Algebra of Observables: Starting from the classical Poisson algebra $\mathcal{O}$ one has to construct the commutator algebra $\mathcal{QO}$, which represents the analogue of $\mathcal{O}$ on the level of quantum theory. The quantum algebra of observables $\mathcal{QO}$ is generated polynomially by a set $\mathcal{Q}\tilde{\mathcal{O}}$ of fundamental observables. The elements of $\mathcal{Q}\tilde{\mathcal{O}}$ are in one-to-one correspondence with the elements of $\tilde{\mathcal{O}}$. The algebraic structure of $\mathcal{QO}$ is defined by the commutators between its elements, which can be obtained derivatively from the commutators between the fundamental observables. The latter have to be inferred from the Poisson brackets between the classical fundamental observables by imposing correspondence and consistency requirements.

  Thus, the observable linear (*i.e.*, Lie) or linearizable symmetries of the system should be preserved upon quantization, *i.e.* this part of the symmetry algebra of the quantum system should be isomorphic to that of the classical system (where the Poisson bracket has to be replaced by $(-i/\hbar)$ times the commutator). For the commutators of arbitrary elements of $\mathcal{Q}\tilde{\mathcal{O}}$ this strict correspondence of commutators and Poisson brackets cannot be required *a priori*. Rather, $(-i/\hbar)$ times the commutators can differ from the Poisson brackets by quantum corrections which are compatible with the correspondence principle. The correction terms must be formed from elements of $\mathcal{QO}$ which possess a well-defined non-vanishing classical limit, multiplied by explicit positive integer powers of $\hbar$. Together with these explicit powers of $\hbar$ they must carry the correct physical dimensions. The covariant transformation properties (with respect to the linear or linearizable part of the symmetry algebra) of the fundamental observables as well as of their commutators should be preserved. If $\tilde{\mathcal{O}}$ carries a gradation or $*$–involution, these structures must also be implemented into $\mathcal{Q}\tilde{\mathcal{O}}$, and the commutator structure must be compatible with them. Of course, if $\tilde{\mathcal{O}}$ is a Lie algebra, this should also be the case for $\mathcal{Q}\tilde{\mathcal{O}}$. Then, $\mathcal{QO}$ is the enveloping algebra of the Lie algebra $\mathcal{Q}\tilde{\mathcal{O}}$.

  This deformation process may not result in the occurrence of additional observables on the level of quantum theory which do not possess a classical analogue.

- Correspondence of Observables: The expressions for specific quantum observables, which correspond to given classical observables, and their commutation relations have to be determined along the same lines.

- The Observable Content of the Constraints: The crucial step is the identification of those Casimir elements of $\mathcal{QO}$ which correspond to the elements of $\mathcal{OC}$ (the principles for their identification are the same as above), and of the conditions which express the observable content of the constraints on the level of quantum theory.

- Identities: In the same way one has to determine the form of those identities for the Casimir elements of $\mathcal{QO}$ which correspond to the classical identities for the Casimir elements of $\mathcal{O}$ which do not involve the constraints.

- Identification of the Physical Representation of $\mathcal{QO}$: Having established the algebraic structure of $\mathcal{QO}$, the physical representation is that irreducible $*$–representation of $\mathcal{QO}$, in which the conditions, which express the observable content of the constraints, and the identities for the remaining Casimir elements of $\mathcal{QO}$ are satisfied.

Note that we do not have to introduce an extended Hilbert space (where the term "extended Hilbert space" refers to any Hilbert space containing unphysical states). Of course, the use of an extended Hilbert space may facilitate the construction of representations of $\mathcal{QO}$. But then the Hilbert space will not be irreducible with respect to $\mathcal{QO}$, and the selection of the physical subspace, *i.e.* of the physical representation of $\mathcal{QO}$, can be carried out with the help of conditions which are imposed on observable quantities.

III. The pseudo-rigid body

In Ref. [1] the pseudo-rigid body (PRB) is defined kinematically by specifying its configuration space. Consider a distribution of mass points in $\mathbf{R}^N$, or a continuous mass distribution, such that the volume (of the convex hull) is non-zero. Let this object undergo collective linear deformations, *i.e.* let each mass point be subject to the same linear transformation. Then, starting from an initial configuration, each other configuration can be obtained by specifying the change in the position of the center of mass, *i.e.* an element of $\mathbf{R}^N$, and in the orientation, shape, and size of the body, *i.e.* an element of $GL_0(N,\mathbf{R})$, the identity component of $GL(N,\mathbf{R})$. That is, the configuration space of the PRB is the group

$$G = GL_0(N,\mathbf{R}) \ltimes \mathbf{R}^N$$

($\ltimes$ denotes the semi-direct product).

Now suppose we are unable to detect different orientations, sizes, and positions of the body, *i.e.* the physical degrees of freedom are the different shapes of the body. Then, the redundant degrees of freedom, namely dilations, rotations, and translations, are described by the action of the gauge group

$$H = \Big(\mathbf{R}^+ \times SO(N)\Big) \ltimes \mathbf{R}^N$$

acting on G from the left. The group H is non-unimodular, the non-unimodularity being effected by the action of the dilations ($\mathbf{R}^+$) on the translations via the semi-direct product structure.

A. Symplectic structure

The configuration space $Q = G$ being an open subset of the space $M(N, \mathbf{R}) \times \mathbf{R}^N$ ($M(N,\mathbf{R})$ is the space of real $(N \times N)$–matrices), we can introduce global coordinates on Q, namely the matrix elements x_{ij}, $1 \le i, j \le N$, of the element $g = (x_{ij}) \in GL_0(N, \mathbf{R})$, and the Cartesian coordinates x_i of the vector $\mathbf{x} \in \mathbf{R}^N$.

The phase space $P = T^*G$ of the system can be identified with the product $G \times \mathcal{L}G^*$ ($\mathcal{L}G^*$ is the dual of the Lie algebra $\mathcal{L}G$ of G) by the trivialization

$$T^* : G \times \mathcal{L}G^* \longrightarrow T^*G, \qquad ((g, \mathbf{x}), (\alpha, \beta)) \longmapsto \mu_{(\alpha,\beta)}(g, \mathbf{x})$$

($(\alpha, \beta) \in \mathcal{L}G^* \simeq M(N, \mathbf{R}) \times \mathbf{R}^N$). In the above coordinates the one-form $\mu_{(\alpha,\beta)} \in T^*G$ is given explicitly by

$$\mu_{(\alpha,\beta)}(g, \mathbf{x}) = \alpha_{ij} dx_{ji} + \beta_i dx_i \tag{3}$$

(repeated indices are summed over).

Denoting the coordinates on $\mathcal{L}G^*$ by (p_{ij}, p_i), the Liouville form can be written as

$$\theta = p_{ij} dx_{ji} + p_i dx_i. \tag{4}$$

The symplectic form is $\omega = -d\theta$.

B. Infinitesimal generators

For the purpose of later reference we will supply the expressions for the infinitesimal generators of the action of the group G on its cotangent bundle T^*G by left and right translations. Let

$$\Phi^L : G \times G \longrightarrow G, \qquad ((h, \mathbf{y}), (g, \mathbf{x})) \longmapsto (hg, \mathbf{y} + h\mathbf{x})$$

be the left action of the group G on itself by left translations, and let $\Phi^{L*} : G \times T^*G \longrightarrow T^*G$ denote the canonical lift of this action to $P = T^*G$. Then, in the above coordinates, the infinitesimal generator for the action of the one-parameter subgroup generated by the element $(A, \mathbf{a}) \in \mathcal{L}G \simeq M(N, \mathbf{R}) \times \mathbf{R}^N$ on P is given by

$$\mathsf{J}^L_{(A,\mathbf{a})}(x, p) = a_{ij} x_{jk} p_{ki} + a_i p_i + p_i a_{ij} x_j. \tag{5}$$

Similarly, let

$$\Phi^R : G \times G \longrightarrow G, \qquad ((h, \mathbf{y}), (g, \mathbf{x})) \longmapsto (gh^{-1}, \mathbf{x} - gh^{-1}\mathbf{y})$$

be the left action of G on itself by right translations, and let Φ^{R*} denote the canonical lift to P. In this case the infinitesimal generator is

$$\mathsf{J}^R_{(A,\mathbf{a})}(x, p) = -x_{ij} a_{jk} p_{ki} - p_i x_{ij} a_j. \tag{6}$$

The Lie algebras which are formed by the elements $\mathsf{J}^{L/R}_{(A,\mathbf{a})}$, $(A,\mathbf{a}) \in \mathcal{L}G$, are isomorphic to $\mathcal{L}G$

$$\{\mathsf{J}^{L/R}_{(A,\mathbf{a})}, \mathsf{J}^{L/R}_{(B,\mathbf{b})}\} = \mathsf{J}^{L/R}_{[(A,\mathbf{a}),(B,\mathbf{b})]} = \mathsf{J}^{L/R}_{([A,B],A\mathbf{b}-B\mathbf{a})}. \tag{7}$$

As left and right translations commute, the corresponding generators satisfy

$$\{\mathsf{J}^{L}_{(A,\mathbf{a})}, \mathsf{J}^{R}_{(B,\mathbf{b})}\} = 0. \tag{8}$$

C. Constraints

Let $D = E_N$ (E_N is the $(N \times N)$–unit matrix), let $\{K_{ij} = -K_{ji}\}$ be the standard basis of $so(N)$, and $\{\mathbf{e}_i\}$ the standard basis of $\mathbf{R}^N$. Then, the elements $(D, 0)$, $(K_{ij}, 0)$, and $(0, \mathbf{e}_i)$ constitute a basis of $\mathcal{L}H$, the Lie algebra of the gauge group H, and the infinitesimal generators

$$\mathsf{D} := \mathsf{J}^L_{(D,0)}, \qquad \mathsf{K}_{ij} := \mathsf{J}^L_{(K_{ij},0)}, \qquad \mathsf{P}_i := \mathsf{J}^L_{(0,\mathbf{e}_i)} \tag{9}$$

span the constraint algebra $\mathcal{C}_0 \simeq \mathcal{L}H$.

D. Fundamental observables

One class of observables, which can readily be obtained, is given by the generators of right translations $\mathsf{J}^R_{(A,\mathbf{a})}$. As the generators $\mathsf{J}^R_{(D,0)} = -\mathsf{J}^L_{(D,0)}$ and $\mathsf{J}^R_{(0,\mathbf{e}_i)} = -\sum_j x_{ji}\mathsf{J}^L_{(0,\mathbf{e}_j)}$ are contained in the intersection $\mathcal{P} \cap \mathcal{C}$ of the strong Poisson commutant of the constraints with the set of weakly vanishing functions, we have to restrict ourselves to the $sl(N, \mathbf{R})$ subalgebra of $\mathcal{L}G$, so that only the observables $\mathsf{J}^R_{(A,0)}$, $A \in sl(N, \mathbf{R})$, can be taken to form part of $\tilde{\mathcal{O}}$.

The action of $SL(N, \mathbf{R})$ has to be supplemented by the "translations along the fibers" generated by an appropriate class of functions on the configuration space $Q = G$ (cf. [10]). These functions can be chosen as

$$\mathsf{X}_{ij} = \mathsf{X}_{ji} = \lambda(\det g)^{-2/N}(g^t g)_{ij} = \lambda(\det g)^{-2/N} x_{ki}x_{kj}, \qquad \lambda > 0 \tag{10}$$

(g^t is the transpose of $g \in GL_0(N, \mathbf{R})$, the physical significance of the parameter λ will be determined in the next section, cf. the discussion below eq. (34)). The functions X_{ij} are obviously invariant under translations ($\mathbf{R}^N$) and under the action of $SO(N)$ from the left. As they are homogeneous of degree zero in x_{ij}, they are also invariant under dilations.

Being the elements of a symmetric matrix, the functions X_{ij} generate an action of the Abelian group $S(N)$ of real symmetric $(N \times N)$–matrices on $P = T^*G$. Choosing the matrices

$$S_{ij} = S_{ji} = \frac{1}{2}(E_{ij} + E_{ji}), \qquad (E_{ij})_{kl} = \delta_{ik}\delta_{jl}$$

as a basis of $S(N)$, the action on P of an element $S = s_{ij}S_{ij} \in S(N)$ $(s_{ij} = s_{ji})$ is given by

$$\mu \longmapsto \mu - s_{ij} d\mathsf{X}_{ij} \tag{11}$$

($\mu \in P$), in coordinates

$$x_{kl} \longmapsto x_{kl}, \qquad p_{kl} \longmapsto p_{kl} - s_{ij}\frac{\partial \mathsf{X}_{ij}}{\partial x_{lk}}. \tag{12}$$

The Poisson brackets of the functions $\mathsf{J}^R_{(A,0)}$ and $\mathsf{S}_{(S)} = s_{ij}\mathsf{X}_{ij}$ close to form a realization of the Lie algebra $sl(n, \mathbf{R}) \ltimes S(N)$. Denoting the elements of $sl(N, \mathbf{R}) \ltimes S(N)$ by (A, S), the infinitesimal generators by $\tilde{\mathsf{J}}_{(A,S)} := \mathsf{J}^R_{(A,0)} + \mathsf{S}_{(S)}$, the commutation relations read

$$\{\tilde{\mathsf{J}}_{(A,S)}, \tilde{\mathsf{J}}_{(B,T)}\} = \tilde{\mathsf{J}}_{[(A,S),(B,T)]} = \tilde{\mathsf{J}}_{([A,B],AT+TA^t-BS-SB^t)}. \tag{13}$$

The corresponding group is $SL(N, \mathbf{R}) \ltimes S(N)$, with the group multiplication law

$$(g, S)(h, T) = (gh, S + gTg^t).$$

This group is also denoted by $CM(N)$, the group of collective motions in N dimensions, and plays a prominent rôle in the description of collective modes of multi-particle systems, *e.g.* in nuclear physics (cf. [12, 13]).

The action of $SL(N, \mathbf{R}) \ltimes S(N)$ on P induces a transitive action on the space of physical states of the system, *i.e.* on the reduced phase space $\bar{P} = T^*(H \setminus G)$. Consequently, the functions $\tilde{\mathsf{J}}_{(A,S)}$ generate the algebra of all observable quantities. Therefore, we shall choose the Lie algebra $cm(N) = sl(N, \mathbf{R}) \ltimes S(N)$ as the Lie algebra $\tilde{\mathcal{O}}$ of fundamental observables. In the next section further justification will be given to this choice.

IV. The cases $N = 2$ and $N = 3$

Having established the kinematical properties of the PRB and the Lie structure of the algebra of fundamental observables, we now have to determine the Casimirs of $\mathcal{O}$ and the observable content of the constraints. This will be done explicitly for the space dimensions $N = 2$ and $N = 3$. For the sake of clarity and simplicity, the case $N = 2$ will be treated in detail, the largely analogous discussion of the case $N = 3$ can then be kept short.

A. $N = 2$

In two space dimensions the configuration space is the group $G = GL_0(2, \mathbf{R}) \ltimes \mathbf{R}^2$, the phase space is $P = T^*G$, and the gauge group $H = (\mathbf{R}^+ \times SO(2)) \ltimes \mathbf{R}^2$. The constraint algebra $\mathcal{C}_0$ is spanned by the functions

$$\mathsf{D} = x_{ij}p_{ji} + x_ip_i, \qquad \mathsf{K} = (x_{1i}p_{i2} - x_{2i}p_{i1}) + (x_1p_2 - x_2p_1), \qquad \mathsf{P}_i = p_i \tag{14}$$

$(i, j \in \{1, 2\})$. Choosing the matrices

$$L_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{15}$$

as a basis of the Lie algebra $sl(2, \mathbf{R})$, the Lie algebra $\tilde{\mathcal{O}} = cm(2) = sl(2, \mathbf{R}) \ltimes S(2)$ is generated by the functions

$$\begin{aligned}
\mathsf{L}_1 &= -\frac{1}{2}(x_{i1}p_{1i} - x_{i2}p_{2i}) && (16)\\
\mathsf{L}_2 &= -\frac{1}{2}(x_{i1}p_{2i} + x_{i2}p_{1i}) && (17)\\
\mathsf{L}_3 &= -\frac{1}{2}(-x_{i1}p_{2i} + x_{i2}p_{1i}) && (18)\\
\mathsf{X}_{ij} &= \frac{\lambda}{\mathsf{d}} x_{ki}x_{kj}, \qquad \mathsf{d} = \det g = \det(x_{ij}). && (19)
\end{aligned}$$

The Lie algebra $cm(2)$ is isomorphic to the Lie algebra $iso(2, 1) = so(2, 1) \ltimes \mathbf{R}^3$ of the Poincaré group in $(2+1)$ dimensions, as can be seen by defining the functions

$$\mathsf{M}_{\mu\nu} := \varepsilon_{\mu\nu\rho}\mathsf{L}_\rho, \quad \mathsf{X}_1 := \mathsf{X}_{12}, \quad \mathsf{X}_2 := \frac{1}{2}(\mathsf{X}_{11} - \mathsf{X}_{22}), \quad \mathsf{X}_3 := \frac{1}{2}(\mathsf{X}_{11} + \mathsf{X}_{22}) \tag{20}$$

$(\mu, \nu, \rho \in \{1, 2, 3\})$ with the commutation relations

$$\{\mathsf{M}_{\mu\nu}, \mathsf{M}_{\mu\rho}\} = g_{\mu\mu}\mathsf{M}_{\nu\rho}, \qquad \{\mathsf{M}_{\mu\nu}, \mathsf{X}_\mu\} = g_{\mu\mu}\mathsf{X}_\nu, \qquad \{\mathsf{X}_\mu, \mathsf{X}_\nu\} = 0 \tag{21}$$

($g_{\mu\nu} = \mathrm{diag}(+ + -)$, the missing commutators vanish or can be obtained from the listed ones using the antisymmetry of $\mathsf{M}_{\mu\nu}$). As the algebra $iso(2, 1)$ is more familiar than the algebra $cm(2)$, we will continue to work with the former. This algebra possesses two Casimir invariants

$$\mathsf{X}^2 = g^{\mu\nu}\mathsf{X}_\mu\mathsf{X}_\nu, \qquad \mathsf{I}_2 = \varepsilon^{\mu\nu\rho}\mathsf{M}_{\mu\nu}\mathsf{X}_\rho \tag{22}$$

(greek indices are raised and lowered with $g^{\mu\nu} = g_{\mu\nu}$, $\varepsilon_{123} = 1$). For $\mathsf{X}^2 \leq 0$ the sign of X_3 is a third invariant. The constraint algebra $\mathcal{C}_0$ does not possess any Casimirs for $N = 2$.

Next we have to determine the identities which express the functional dependencies between the Casimirs of $\mathcal{O}$ and the constraints, and the identities which are fulfilled by the Casimirs of $\mathcal{O}$ without involving the constraints. We find

$$\lambda\bar{\mathsf{K}} \equiv \mathsf{I}_2, \qquad \bar{\mathsf{K}} := \mathsf{K} - x_1\mathsf{P}_2 + x_2\mathsf{P}_1 \tag{23}$$

and

$$\mathsf{X}^2 \equiv -\lambda^2, \qquad \mathrm{sign}(\mathsf{X}_3) = 1. \tag{24}$$

The function $\bar{\mathsf{K}}$, which represents the observable content of the constraints, is not a Casimir of $\mathcal{C}_0$ but a generalized Casimir element of the constraints. This property is not an effect of the non-unimodularity of the gauge group H, but simply of its semi-direct product structure (which, in turn, causes also the non-unimodularity).

It should be noted that the constraint algebra $\mathcal{C}_0$, which is a semi-direct product, can be replaced by an equivalent constraint algebra $\bar{\mathcal{C}}_0 = (\mathbf{R}^+ \times so(2)) \times \mathbf{R}^2$, which is a direct product and is generated by $\bar{\mathsf{K}}$, $\bar{\mathsf{D}} = \mathsf{D} - x_i \mathsf{P}_i$, and $\bar{\mathsf{P}}_i = \mathsf{P}_i$ (for $N = 2$, $\bar{\mathcal{C}}_0$ is even Abelian). $\bar{\mathsf{K}}$ is a Casimir of $\bar{\mathcal{C}}_0$, that is it fulfills the definition of an element of $\mathcal{OC}$ as it was given in Ref. [2].

It should also be noted that the dilation constraint D does not contribute to the observable content of the constraints, *i.e.* it cannot have any impact on the selection of the physical representation of the algebra of observables. Therefore it would be quite unphysical to make the quantization of the system depend on requirements which are imposed on the quantization of this quantity which, from the point of view of the constrained system, is unobservable. It is one of the merits of Ref. [1] to have shown that a naive application of the Dirac quantization scheme, which requires the quantization of this quantity, can lead to substantially wrong results: it is the requirement that the operator corresponding to D upon quantization be formally self-adjoint with respect to an inner product on an extended Hilbert space containing unphysical states, which necessitates the introduction of the correction term (2).

Of course, from the discussion of this single example nothing can be said about the general case of a non-unimodular gauge group.

B. Dynamics

In this paragraph it will be shown how the Hamiltonian for the free dynamics of the constrained system can be expressed as a function of the $cm(2)$ observables, and how these observables can be given a physical interpretation.

Let the body be composed of $n \geq 3$ individual mass points with the same mass m, not all of them lying on the same line. Let the positions of the mass points, relative to the center of mass, at the time $t = 0$ be $\mathbf{x}_a^0$, $a = 1, 2, \ldots, n$. Then the positions at time $t > 0$, implementing the condition that all mass points be subject to the same linear transformation, can be written as

$$\mathbf{x}_a(t) = g(t)\mathbf{x}_a^0, \qquad g(t) \in GL_0(2, \mathbf{R}), \quad g(0) = E_2. \tag{25}$$

Define the mass–quadrupole tensor $q(t)$ by

$$q_{ij}(t) = \sum_{a=1}^{n} m x_{ai}(t) x_{aj}(t) = (g(t) q(0) g^t(t))_{ij} \tag{26}$$

and choose the basis in the center of mass frame such that $q^0 := q(0)$ becomes diagonal: $q^0 = \mathrm{diag}(q_1, q_2)$. The kinetic energy of the unconstrained motion of the

body, relative to the center of mass, can be expressed by q^0 and the time derivative $\dot{g}(t)$ of $g(t)$

$$\mathsf{T} = \frac{1}{2}\mathrm{tr}(\dot{g}(t)q^0\dot{g}^t(t)). \tag{27}$$

Now, using a trivialization analogous to (3) for the tangent bundle

$$T : G \times \mathcal{L}G \longrightarrow TG, \qquad ((g,\mathbf{x}),(A,\mathbf{a})) \longmapsto v_{(A,\mathbf{a})}(g,\mathbf{x}) = a_{ij}\frac{\partial}{\partial x_{ij}} + a_i\frac{\partial}{\partial x_i}$$

($A = (a_{ij}) \in gl(N,\mathbf{R}) \simeq M(N,\mathbf{R})$, $\mathbf{a} \in \mathbf{R}^N$) and neglecting the translations, we can identify the tangent vector $\dot{g}(t)$ with an element $\Omega(t)$ of the Lie algebra $gl(2,\mathbf{R})$, which has the same components as $\dot{g}(t)$. Expanding Ω into the basis $D = E_2$ and $I_\mu := L_\mu$ (eq. 15) of $gl(2,\mathbf{R})$

$$\Omega = \frac{1}{2}\omega_D D + \omega_\mu I_\mu \tag{28}$$

the kinetic energy becomes a function of the generalized "angular" velocities ω_D and ω_μ: $\mathsf{T} = \mathsf{T}(\omega_D, \omega_\mu)$. The generalized "angular" momenta conjugate to ω_D and ω_μ

$$\mathsf{J}_D = \frac{\partial \mathsf{T}}{\partial \omega_D}, \qquad \mathsf{J}_\mu = \frac{\partial \mathsf{T}}{\partial \omega_\mu} \tag{29}$$

are the infinitesimal generators of the group action by left translations, *i.e.* in the above coordinates on the group

$$\mathsf{J}_D = \frac{1}{2}\bar{\mathsf{J}}^L_{(D,0)}, \qquad \mathsf{J}_\mu = \bar{\mathsf{J}}^L_{(I_\mu,0)} \tag{30}$$

where $\bar{\mathsf{J}}^L_{(A,0)} = a_{ij}x_{jk}p_{ki}$ (this is the same as (5), neglecting the translations). The canonical Hamiltonian for the free motion of the unconstrained system is a function of the momenta J_D and J_μ

$$\mathsf{H}_0 = \omega_D\mathsf{J}_D + \omega_\mu\mathsf{J}_\mu - \mathsf{T} = \frac{(\mathsf{J}_D + \mathsf{J}_1)^2 + (\mathsf{J}_2 + \mathsf{J}_3)^2}{2q_1} + \frac{(\mathsf{J}_D - \mathsf{J}_1)^2 + (\mathsf{J}_2 - \mathsf{J}_3)^2}{2q_2}. \tag{31}$$

Now we have to implement the constraints such that different points on the same gauge orbit are dynamically identified. In accordance with Dirac [6] this can be done by adding to H_0 a combination of the constraints with arbitrary functions as coefficients. Again neglecting the translations, we obtain the extended Hamiltonian

$$\mathsf{H}_E = \mathsf{H}_0 + \lambda_D\bar{\mathsf{D}} + \lambda_K\bar{\mathsf{K}} \tag{32}$$

($\bar{\mathsf{D}}$ and $\bar{\mathsf{K}}$ as above, note that $\bar{\mathsf{D}} = 2\mathsf{J}_D$, $\bar{\mathsf{K}} = 2\mathsf{J}_3$). Making use of the arbitrariness of the multipliers λ_D and λ_K, H_E can be brought into the form

$$\mathsf{H}_E = \mathsf{H} + \kappa_D\bar{\mathsf{D}} + \kappa_K\bar{\mathsf{K}}, \qquad \mathsf{H} = \frac{1}{2\theta}\mathsf{J}_\mu\mathsf{J}^\mu, \qquad \theta := \frac{q_1q_2}{q_1+q_2} \tag{33}$$

where κ_D and κ_K are still arbitrary, and $\mathsf{J}^2 = \mathsf{J}_\mu \mathsf{J}^\mu$ is the quadratic Casimir of the $sl(2, \mathbf{R})$ subalgebra of $gl(2, \mathbf{R})$. Finally, observing that we have the identity $\mathsf{J}^2 = \mathsf{L}^2$, where $\mathsf{L}^2 = \mathsf{L}_\mu \mathsf{L}^\mu$ is the quadratic Casimir of the $sl(2, \mathbf{R}) \simeq so(2,1)$ subalgebra of the $cm(2) \simeq iso(2,1)$ algebra, the Hamiltonian H can be expressed as a function of observables

$$\mathsf{H} = \frac{1}{2\theta} \mathsf{L}^2. \tag{34}$$

The symmetry algebra of H, *i.e.* of the free motion of the constrained system, is just the $sl(2, \mathbf{R}) \simeq so(2,1)$ subalgebra of $cm(2)$. For other forms of collective Hamiltonians (especially for $cm(3)$) cf. the cited literature [12, 13] and Refs. therein.

The fact that the symmetry algebra of the free motion is already included in the algebra $cm(2)$ and that the Hamiltonian is a polynomial function of its generators provides a dynamical justification for the choice of the algebra $cm(2)$ as the Lie algebra $\tilde{\mathcal{O}}$ of fundamental observables. Furthermore, the functions X_{ij} can also be given a physical interpretation. They can be thought of as (redundant) coordinates on the orbit through the point $\lambda E_2 \in S(2)$ under the right action $S \longmapsto g^t S g$ of $SL(2, \mathbf{R})$ on $S(2)$. This orbit consists of all positive definite symmetric (2×2)–matrices with determinant λ^2, and, with $\lambda = \sqrt{\det q^0}$, can be identified with the space of all mass–quadrupole tensors of determinant $q_1 q_2$. Moreover, as the quantity $\sqrt{q_1 q_2}$ is equal to the product of the mass and the volume of the body, the invariant $\mathsf{X}^2 = -\det(\mathsf{X}_{ij})$ can be interpreted as measuring the latter (cf. [13]).

C. $N = 3$

The discussion of the three-dimensional case proceeds along the same lines as that of the two-dimensional one. We will, therefore, restrict ourselves to enumerating the relevant points.

Constraints

A basis of the Lie algebra $\mathcal{L}H$ of the gauge group $H = (\mathbf{R}^+ \times SO(3)) \ltimes \mathbf{R}^3$ can be taken to consist of the elements $(D, 0)$, $(K_i, 0)$ and $(0, \mathbf{e}_i)$, $i = 1, 2, 3$, where $D = E_3$ and $\{K_i | (K_i)_{jk} = -\varepsilon_{ijk}\}$ is the standard basis of $so(3)$. The corresponding generators of the constraint algebra $\mathcal{C}_0$ are

$$\mathsf{D} = x_{ij} p_{ji} + x_i p_i, \qquad \mathsf{K}_i = \varepsilon_{ijk}(- x_{kl} p_{lj} + x_j p_k), \qquad \mathsf{P}_i = p_i. \tag{35}$$

Observables

For the $sl(3,\mathbf{R})$ subalgebra of the Lie algebra $cm(3) = sl(3,\mathbf{R}) \ltimes S(3)$ it is convenient to choose the redundant basis $(T_{ij}, 0)$, where

$$T_{ij} = E_{ij} - \frac{1}{3}\delta_{ij}E_3, \qquad \sum_{i=1}^{3} T_{ii} = 0. \tag{36}$$

The basis for $S(3)$ can be chosen as above: $S_{ij} = S_{ji} = \frac{1}{2}(E_{ij} + E_{ji})$. The corresponding generators of the algebra $\tilde{\mathcal{O}}$ of fundamental observables are

$$\mathsf{J}_{ij} = \mathsf{J}^R_{(T_{ij},0)} = -x_{kl}(T_{ij})_{lm}p_{mk}, \qquad \mathsf{X}_{ij} = \frac{\lambda}{(\det g)^{2/3}}x_{ki}x_{kj}, \tag{37}$$

their commutation relations read

$$\{\mathsf{J}_{ij}, \mathsf{J}_{kl}\} = \delta_{jk}\mathsf{J}_{il} - \delta_{il}\mathsf{J}_{kj} \tag{38}$$

$$\{\mathsf{J}_{ij}, \mathsf{X}_{kl}\} = \delta_{jk}\mathsf{X}_{il} + \delta_{jl}\mathsf{X}_{ik} - \frac{2}{3}\delta_{ij}\mathsf{X}_{kl} \tag{39}$$

$$\{\mathsf{X}_{ij}, \mathsf{X}_{kl}\} = 0. \tag{40}$$

Casimirs of $\mathcal{O}$

The algebra $cm(3)$ possesses two polynomial Casimir invariants (cf. [13])

$$\mathsf{I}_3 := \det(\mathsf{X}_{ij}), \qquad \mathsf{I}_5 := -\frac{1}{2}\mathsf{V}_k\bar{\mathsf{V}}_k \tag{41}$$

where

$$\mathsf{V}_k = \varepsilon_{kij}\bar{\mathsf{X}}_{li}\mathsf{J}_{lj}, \qquad \bar{\mathsf{V}}_k = \varepsilon_{kij}\mathsf{X}_{il}\mathsf{J}_{jl}, \qquad \bar{\mathsf{X}}_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}\mathsf{X}_{km}\mathsf{X}_{ln} = \bar{\mathsf{X}}_{ji}. \tag{42}$$

A third invariant is the signature $\mathrm{Sig}(\mathsf{X})$ of the matrix $\mathsf{X} = (\mathsf{X}_{ij})$, which is defined as twice the number of positive eigenvalues minus the rank of X.

Identities and the observable content of the constraints

There are two identities for the Casimirs of $\mathcal{O}$, which do not involve the constraints

$$\mathsf{I}_3 = \det \mathsf{X} \equiv \lambda^3, \qquad \mathrm{Sig}(\mathsf{X}) = 3 \tag{43}$$

(*i.e.* X is positive definite), and one functional identity which determines the observable content of the constraints

$$\mathsf{I}_3\Lambda = \lambda^3\Lambda \equiv \mathsf{I}_5, \qquad \Lambda = \sum_{i=1}^{3}\bar{\mathsf{K}}_i^2 \tag{44}$$

where

$$\bar{\mathsf{K}}_i = \mathsf{K}_i - \varepsilon_{ijk} x_j \mathsf{P}_k. \tag{45}$$

Again, Λ is not a Casimir of $\mathcal{C}_0$ but a generalized Casimir element of the constraints. Going over to the equivalent constraint algebra $\bar{\mathcal{C}}_0 = (\mathbf{R}^+ \times SO(3)) \times \mathbf{R}^3$ generated by $\bar{\mathsf{D}} = \mathsf{D} - x_i \mathsf{P}_i$, $\bar{\mathsf{K}}_i$ and $\bar{\mathsf{P}}_i = \mathsf{P}_i$, Λ becomes a Casimir of $\bar{\mathcal{C}}_0$.

Note that, under the additional conditions expressed by the identities (43), the Casimir I_5 is always a non-negative function. This can be seen as follows. In every Hamiltonian action (cf. [14]) of the Lie algebra $cm(3)$ on a symplectic manifold M the generators of this action are uniquely defined – not only up to a constant – because the first and second cohomology groups of $cm(3)$ vanish [15]. Therefore, the matrix X of the generators X_{ij} of the $S(3)$ subalgebra assumes the value $\mathsf{X} = \lambda E_3$ (*i.e.* there is a point $m \in M$, such that $\mathsf{X}(m) = \lambda E_3$), given that the generators fulfill the identities (43). For $\mathsf{X} = \lambda E_3$ we have $\bar{\mathsf{X}} = 2\lambda^2 E_3$, and, at the point m, I_5 can be written as

$$\mathsf{I}_5(m) = \lambda^3 \sum_{i=1}^{3} \mathsf{N}_i^2(m), \qquad \mathsf{N}_i = \varepsilon_{ijk} \mathsf{J}_{jk}. \tag{46}$$

But I_5 is constant on the orbit through m, and so I_5 is non-negative everywhere because M is foliated by the orbits of the $cm(3)$ action.

In the case at hand I_5 assumes the value zero, the minimum value which is compatible with the identities (43), on the constraint surface. That is, the effect, on the observable sector of the system, of the vanishing of the constraints consists in restricting the observable $\lambda^3\Lambda = \mathsf{I}_5$ to its minimum. Consequently, the observable content of the constraints, *i.e.*, the condition which is imposed on the Casimir I_5 of the algebra of observables by the vanishing of the constraints via the identity (44), can be given two equivalent formulations, a numerical and an algebraic one. The numerical formulation states that the Casimir $\lambda^3\Lambda = \mathsf{I}_5$ assumes the *value* zero, whereas the algebraic formulation states that it has to assume the *minimum* possible value compatible with the identities (43).

Dynamics

In the same way as in the two-dimensional case the Hamiltonian for the free constrained dynamics can be expressed as a function of the $cm(3)$ generators. For $q^0 = qE_3$ (which is not a restriction because it can always be achieved by an $SL(3,\mathbf{R})$–transformation, *i.e.* by a change of basis) it is proportional to the quadratic Casimir of the $sl(3,\mathbf{R})$ subalgebra

$$\mathsf{H} = \frac{1}{2q} g_{ijkl} \mathsf{J}_{ji} \mathsf{J}_{lk}, \qquad g_{ijkl} = \mathrm{tr}(T_{ij} T_{kl}). \tag{47}$$

V. Quantization of the PRB

In this section we will carry out the quantization of the PRB in the two- and three-dimensional case, following the quantization scheme outlined in Sec. II. In both cases we end up with a unique identification of the physical representation of the quantum algebra of observables.

A. $N = 2$

The first step of the quantization consists in the construction of the Lie algebra of fundamental observables $\mathcal{Q}\tilde{\mathcal{O}}$ which corresponds to the classical algebra $\tilde{\mathcal{O}}$ on the level of quantum theory. The only quantum correction of the classical commutation relations, compatible with the principles formulated in Sec. II, would be a central extension of the algebra $iso(2,1)$. However, the Lie algebra $iso(2,1)$ does not possess any non-trivial central extension, because its second cohomology vanishes (cf. [14]). Thus, the Lie algebraic structure of the algebra of fundamental observables remains unchanged, *i.e.* the algebra $\mathcal{Q}\tilde{\mathcal{O}}$, as a commutator algebra, is isomorphic to the Lie algebra $iso(2,1)$, and the algebra of observables $\mathcal{Q}\mathcal{O}$ is isomorphic to its enveloping algebra. The commutation relations of the generators $\hat{\mathsf{M}}_{\mu\nu}$ and $\hat{\mathsf{X}}_\mu$ of $\mathcal{Q}\tilde{\mathcal{O}}$ read

$$[\hat{\mathsf{M}}_{\mu\nu}, \hat{\mathsf{M}}_{\mu\rho}] = i\hbar g_{\mu\mu}\hat{\mathsf{M}}_{\nu\rho}, \qquad [\hat{\mathsf{M}}_{\mu\nu}, \hat{\mathsf{X}}_{\mu}] = i\hbar g_{\mu\mu}\hat{\mathsf{X}}_{\nu}, \qquad [\hat{\mathsf{X}}_{\mu}, \hat{\mathsf{X}}_{\nu}] = 0. \tag{48}$$

The expressions for the Casimirs of $\mathcal{Q}\mathcal{O}$ in terms of $\hat{\mathsf{M}}_{\mu\nu}$ and $\hat{\mathsf{X}}_\mu$ are the same as in eqs. (22). Observe that there are no factor ordering ambiguities in the definition of the Casimirs. Of course, the expressions for other observables, in terms of the basic $iso(2,1)$ observables, and their commutation relations can still acquire quantum corrections.

Next we have to determine the form which the identities (24) for the Casimirs of $iso(2,1)$ take upon quantization. Imposing the requirements that the classical identities be reproduced in the classical limit, that the possible correction terms must carry explicit positive integer powers of $\hbar$ and the correct overall physical dimensions, and that the identities may only involve Casimirs, it can easily be seen that there are no quantum corrections available. That is, the form of the identities remains unchanged

$$\hat{\mathsf{X}}^2 \equiv -\lambda^2, \qquad \mathrm{sign}(\hat{\mathsf{X}}_3) = 1 \tag{49}$$

(the latter identity means that $\hat{\mathsf{X}}_3$ is a positive operator).

The crucial step is the identification of that Casimir element in the algebra of observables, which corresponds to the classical observable $\bar{\mathsf{K}}$. Applying the same principles as above, it can be seen that the only possible correction of the classical expression is a constant term

$$\lambda\hat{\bar{\mathsf{K}}} = \hat{\mathsf{I}}_2 + \hbar\lambda c, \qquad c = \text{const.} \tag{50}$$

This constant contribution can be excluded by an additional discrete symmetry (parity) of the classical system. Implementing it as a reflection symmetry, the cooordinates transform as

$$(x_i, p_i) \longrightarrow (-1)^i (x_i, p_i), \qquad (x_{ij}, p_{ij}) \longrightarrow (-1)^{i+j} (x_{ij}, p_{ij}). \tag{51}$$

The classical observables I_2 and $\bar{\mathsf{K}}$ transform as pseudoscalars. If we require this symmetry to be realized in the quantum theory, the constant c must vanish. Therefore, we have the identification

$$\lambda \hat{\bar{\mathsf{K}}} = \hat{\mathsf{I}}_2, \tag{52}$$

and the observable content of the constraints is expressed by the induced operator identity $\hat{\mathsf{I}}_2 = 0$.

Identification of the physical representation of $\mathcal{QO}$

As the Hermitian irreducible representations (HIR) of the Lie algebra $iso(2,1)$ can be obtained from the unitary irreducible representations (UIR) of the group $ISO(2,1)$, we will in the sequel determine the physical representation of the latter. This can most easily be done by using the method of induced representations (cf. [16]).

First, the identities (49) fix an orbit under the action of $SO(2,1)$ on the dual space of the space of characters of the Abelian subgroup $\mathbf{R}^3$. This orbit can be identified with the "mass" shell

$$H^+_{2,1}(\lambda) = \left\{ \mathbf{x} \in \mathbf{R}^3 \middle| \mathbf{x}^2 = g^{\mu\nu} x_\mu x_\nu = -\lambda^2,\ x_3 > 0 \right\} \tag{53}$$

and is isomorphic to the homogeneous space $SO(2)\backslash SO(2,1)$. This means that the identities (49) single out a class of representations of $ISO(2,1)$, namely those representations which are induced by the UIR of the subgroup $SO(2) \ltimes \mathbf{R}^3$ associated with the orbit $H^+_{2,1}(\lambda)$. The individual representations in this class are characterized by the corresponding representation of the little group $SO(2)$, labelled by an integer $k_0 \in \mathbf{Z}$. They will be denoted $\mathrm{D}_2(\lambda, k_0)$.

The eigenvalue of the Casimir $\hat{\mathsf{I}}_2$ in the representation $\mathrm{D}_2(\lambda, k_0)$ is given by $\hat{\mathsf{I}}_2 = \hbar\lambda k_0$. Consequently, the identity $\hat{\mathsf{I}}_2 = \lambda\hat{\bar{\mathsf{K}}} = 0$, acting as a representation condition, uniquely selects the representation $\mathrm{D}_2(\lambda, 0)$ as the physical representation of the group $ISO(2,1)$ and of the algebra $\hat{\mathcal{QO}}$ of fundamental observables.

Finally, we want to give an explicit description of the physical representation of the group $ISO(2,1)$, respectively of the equivalent representation of $CM(2)$, and to determine the spectrum of the free Hamiltonian.

The UIR of $ISO(2,1)$ which is induced by the spin-zero representation of the subgroup $SO(2) \ltimes \mathbf{R}^3$ associated with the orbit $H^+_{2,1}(\lambda)$ is realized on the Hilbert space

$$\mathcal{H}_{2,1} = L^2(H^+_{2,1}(\lambda), d\mu(x)), \qquad d\mu(x) = \delta(\mathbf{x}^2 + \lambda^2)\theta(x_3) d^3x. \tag{54}$$

An element $(g, \mathbf{a}) \in ISO(2,1)$ acts on $\mathcal{H}_{2,1}$ via the unitary operator $\hat{\mathsf{U}}(g, \mathbf{a})$

$$(\hat{\mathsf{U}}(g, \mathbf{a})\Psi)(\mathbf{x}) = e^{\frac{i}{\hbar}\mathbf{a}\cdot\mathbf{x}}\Psi(g^{-1}\mathbf{x}) \tag{55}$$

($\mathbf{a} \cdot \mathbf{x} = x_\mu a^\mu$). This representation is unitarily equivalent to the following UIR of $CM(2)$, induced by the spin-zero representation of the subgroup $SO(2) \ltimes S(2)$ related to the orbit

$$O_2(\lambda) = \Big\{X \in S(2) \Big| \det X = \lambda^2,\ X \text{ positive definite}\Big\},$$

realized on the Hilbert space

$$\mathcal{H}_2 = L^2(O_2(\lambda), d\mu(X))$$

where

$$d\mu(X) = \delta(\det X - \lambda^2)\theta(X) dX_{11} dX_{12} dX_{22} \tag{56}$$

$$\theta(X) = \begin{cases} 1 & \text{if } X \text{ is positive definite} \\ 0 & \text{else.} \end{cases} \tag{57}$$

The action of an element $(g, S) \in CM(2)$ on $\mathcal{H}_2$ is given by

$$(\hat{\mathsf{U}}(g, S)\Psi)(X) = e^{\frac{i}{\hbar}\operatorname{tr}(SX)}\Psi(g^t X g). \tag{58}$$

Upon restriction to the subgroup $SO(2,1)$ the above representation of $ISO(2,1)$ decomposes into a direct integral

$$\int_0^\infty D(\sigma) d\mu(\sigma)$$

of the representations $D(\sigma)$ of the continuous series of UIR of $SO(2,1)$ (cf. [17]). The parameter σ which characterizes the representations is connected to the eigenvalues of the quadratic Casimir $\hat{\mathsf{L}}^2$ of $so(2,1)$ in these representations via

$$\operatorname{spec}(\hat{\mathsf{L}}^2) = \Big\{\hbar^2(\frac{1}{4} + \sigma^2)\Big|\sigma \in \mathbf{R}^+\Big\}. \tag{59}$$

Therefore, assuming that the classical relation (34) between the free Hamiltonian and L^2 remains unchanged because there are no factor ordering problems for $\hat{\mathsf{H}}$ in terms of the fundamental observables $\hat{\mathsf{L}}_\mu$ (a constant quantum correction $\hbar^2 \frac{c}{2\theta}$ is compatible with the correspondence principle, but it cannot be measured), the spectrum of $\hat{\mathsf{H}}$ is given by

$$\operatorname{spec}(\hat{\mathsf{H}}) = \Big\{\frac{1}{2\theta}\hbar^2(\frac{1}{4} + \sigma^2)\Big|\sigma > 0\Big\}. \tag{60}$$

B. $N = 3$

Applying the same principles and argumentation as in the two-dimensional case, the Lie algebra $\mathcal{Q}\tilde{\mathcal{O}}$ has to be taken to be isomorphic to the Lie algebra $cm(3)$. The commutation relations of the generators $\hat{\mathsf{J}}_{ij}$ and $\hat{\mathsf{X}}_{kl}$ read

$$[\hat{\mathsf{J}}_{ij}, \hat{\mathsf{J}}_{kl}] = i\hbar(\delta_{jk}\hat{\mathsf{J}}_{il} - \delta_{il}\hat{\mathsf{J}}_{kj}) \tag{61}$$

$$[\hat{\mathsf{J}}_{ij}, \hat{\mathsf{X}}_{kl}] = i\hbar(\delta_{jk}\hat{\mathsf{X}}_{il} + \delta_{jl}\hat{\mathsf{X}}_{ik} - \frac{2}{3}\delta_{ij}\hat{\mathsf{X}}_{kl}) \tag{62}$$

$$[\hat{\mathsf{X}}_{ij}, \hat{\mathsf{X}}_{kl}] = 0. \tag{63}$$

The expressions for the Casimirs of $cm(3)$ in terms of the generators $\hat{\mathsf{J}}_{ij}$ and $\hat{\mathsf{X}}_{ij}$ are the same as the classical ones (eq. (41)). Note that, because of the identities

$$\hat{\mathsf{V}}_k = \varepsilon_{kij}\hat{\hat{\mathsf{X}}}_{li}\hat{\mathsf{J}}_{lj} = \varepsilon_{kij}\hat{\mathsf{J}}_{lj}\hat{\hat{\mathsf{X}}}_{li}, \qquad \hat{\hat{\mathsf{V}}}_k = \varepsilon_{kij}\hat{\mathsf{X}}_{il}\hat{\mathsf{J}}_{jl} = \varepsilon_{kij}\hat{\mathsf{J}}_{jl}\hat{\mathsf{X}}_{il}, \qquad \hat{\mathsf{V}}_k\hat{\hat{\mathsf{V}}}_k = \hat{\hat{\mathsf{V}}}_k\hat{\mathsf{V}}_k,$$

the Casimir $\hat{\mathsf{I}}_5 = -\frac{1}{2}\hat{\mathsf{V}}_k\hat{\hat{\mathsf{V}}}_k$ is unambiguously defined by its classical expression. The form of the identities (43) remains unchanged

$$\hat{\mathsf{I}}_3 = \det\hat{\mathsf{X}} \equiv \lambda^3, \qquad \mathrm{Sig}(\hat{\mathsf{X}}) = 3 \tag{64}$$

and the identification of the Casimir element expressing the observable content of the constraints can be accomplished up to a contribution proportional to $\hat{\mathsf{I}}_3$

$$\hat{\mathsf{I}}_3\hat{\Lambda} = \lambda^3\hat{\Lambda} = \hat{\mathsf{I}}_5 + c\hbar^2\hat{\mathsf{I}}_3 = \hat{\mathsf{I}}_5 + c\hbar^2\lambda^3, \qquad c = \text{const.} \tag{65}$$

The identities (64) determine a class of UIR of the group $CM(3)$. These representations are induced by the UIR of the subgroup $SO(3) \ltimes S(3)$ associated with the orbit

$$O_3(\lambda) = \left\{X \in S(3)\middle|\det X = \lambda^3,\ X \text{ positive definite}\right\}.$$

$O_3(\lambda)$ is isomorphic to the homogeneous space $(SO(3) \ltimes S(3))\backslash CM(3)$. The representations are labelled by a discrete parameter j, $2j \in \mathbf{N}_0$, which characterizes the corresponding representation of the little group $SO(3)$ (respectively of its covering group $SU(2)$). They will be denoted $\mathrm{D}_3(\lambda, j)$. The eigenvalues of the Casimir $\hat{\mathsf{I}}_5$ in these representations are given by (cf. [13])

$$\hat{\mathsf{I}}_5 = \hbar^2\lambda^3 j(j+1). \tag{66}$$

Applying the algebraic formulation of the observable content of the constraints given in Sec. IV.C., the implementation of the constraints can be carried out without explicitly having to determine the constant c in eq. (65). Accordingly, the physical representation is distinguished by the fact that the observable $\hat{\Lambda}$ assumes the minimum value compatible with the identities (64). As in the classical case,

under the additional conditions expressed by (64), the Casimir $\hat{\mathsf{I}}_5$ is non-negative. It assumes its minimum value zero in the representation $\mathrm{D}_3(\lambda, 0)$. Therefore, irrespective of the value of the constant c, the physical representation of $CM(3)$ can be identified uniquely as the representation $\mathrm{D}_3(\lambda, 0)$. For $c = 0$, $\hat{\Lambda}$ assumes the value zero in this representation.

The physical representation is realized on the Hilbert space

$$\mathcal{H}_3 = L^2(O_3(\lambda),\ d\mu(X)), \quad d\mu(X) = \delta(\det X - \lambda^3)\theta(X) \prod_{i \leq j} dX_{ij} \tag{67}$$

with the group action being given by $(g, S) \longmapsto \hat{\mathsf{U}}(g, S)$

$$(\hat{\mathsf{U}}(g, S)\Psi)(X) = e^{\frac{i}{\hbar}\operatorname{tr}(SX)}\Psi(g^t X g). \tag{68}$$

As a representation space for $CM(3)$, the Hilbert space $\mathcal{H}_3$ coincides with the one that has been determined in Ref. [1].

Upon restriction to $SL(3, \mathbf{R})$ the representation $\mathrm{D}_3(\lambda, 0)$ decomposes into a direct integral of UIR of $SL(3, \mathbf{R})$. This direct integral decomposition yields the spectral resolution of the Hamiltonian $\hat{\mathsf{H}}$, which (again up to an overall additive constant $\hbar^2 \frac{c}{2q}$) can be identified as being proportional to the second order Casimir of $sl(3, \mathbf{R})$

$$\hat{\mathsf{H}} = \frac{1}{2q} g_{ijkl} \hat{\mathsf{J}}_{ji} \hat{\mathsf{J}}_{lk}$$

(because of the identity $g_{ijkl}\hat{\mathsf{J}}_{ji}\hat{\mathsf{J}}_{lk} = g_{ijkl}\hat{\mathsf{J}}_{lk}\hat{\mathsf{J}}_{ji}$ there are no factor ordering ambiguities). Unfortunately, the required decomposition and the corresponding eigenvalues of the Casimir operators are not available in the standard mathematical literature on the subject, so that we cannot determine the spectrum of the Hamiltonian explicitly.

VI. Conclusions

In this paper I have demonstrated the usefulness and effectiveness of our algebraic constraint quantization scheme for the construction of the quantum theory of a system with a complicated, non-Abelian and non-unimodular, gauge group. By the discussion of this example the algebraic method could be given a more precise and more widely applicable formulation.

I would like to stress the importance of the proper identification of the observable content of the constraints. It allows to carry over the implementation of the constraints to the observable sector and to deal only with observable quantities. Thus it can be avoided to make the quantization of a constrained system depend on manipulations in the unobservable sector of the constraints. Once the observable content of the constraints has been determined classically, the constraints are discarded and the construction of the quantum theory proceeds by applying correspondence and consistency requirements to observable quantities. And, after all, these principles can only be applied to observable quantities. There are no guiding principles for the construction of a quantum theory of unobservable quantities.

In the case of the pseudo-rigid body this feature of the algebraic constraint quantization scheme leads to a considerable simplification, in that it is not necessary to take care of the non-unimodularity of the gauge group, caused by the presence of the dilation constraint. In the conventional approaches to the quantization of constrained systems this structure of the gauge group leads to serious difficulties.

Future applications of our algebraic constraint quantization scheme will be concerned with the proper implementation of the constraints – and not of gauge conditions – of physically relevant relativistic field theories into the respective quantum theories.

References

[*] The contents of the present paper and that of Ref. [2] has been submitted as the author's inaugural thesis:
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[1] C. Duval, J. Elhadad, M.J. Gotay, and G.M. Tuynman, Nonunimodularity and the quantization of the pseudo-rigid body, in *Hamiltonian Systems, Transformation Groups and Spectral Transform Methods*, eds. J. Harnad and J.E. Marsden (Les Publications du CRM, Montréal, 1990); CNRS preprint CPT-90/P.2358.

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[5] In the sequel the term "constraints" will be used for the generators of infinitesimal gauge transformations as well as for the constraining conditions themselves. Its precise meaning will be clear from the context. We will only consider systems with *first class* constraints [6].

[6] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).

[7] Where the Casimir invariants, or simply Casimirs, comprise the proper Casimir operators, *i.e.* the generators of the center of the algebra of observables, as well as all further invariant quantities which can be used for the characterization of (classes of) irreducible representations, like, *e.g.*, the sign of the energy in the physical representations of the Poincaré algebra.

[8] As in the case of the constraints, this term will be used not only for the conditions themselves, but also for the gauge invariant combinations of the constraints, respectively the corresponding Casimir elements of the algebra of observables, which enter into the functional dependencies.

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[11] In some cases it may be more favourable that not all the generators of $\mathcal{S}$ are included in $\tilde{\mathcal{O}}$ but, *e.g.*, only the generators of the space–time symmetries. In that case those elements of $\mathcal{S}$ which are not contained in $\tilde{\mathcal{O}}$ must be simple polynomial functions in the elements of $\tilde{\mathcal{O}}$ and $\mathcal{S}$ must be treated separately in the same way as $\tilde{\mathcal{O}}$ respectively $\mathcal{O}$ (*cf* Ref. [2]).

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